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BETA-ENERGY AVERAGING AND BETA SPECTRA

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M. G. Stamatelatos and T. R. England

ABSTRACT

A simple yet highly accurate method for approximately calculating spectrum-averaged beta energies and beta spectra for radioactive nuclei is presented. This method should prove useful for users who wish to obtain accurate answers without complicated calculations of Fermi functions, complex gamma functions, and time-consuming numerical integrations as required by the more exact theoretical expressions. Therefore, this method should be a good time-saving alternative for investigators who need to make calculations involving large numbers of nuclei (e.g., fission products) as well as for occasional users interested in a restricted number of nuclides. The average beta-energy values calculated by this method differ from those calculated by "exact" methods by no more than 1% for nuclides with atomic numbers in the 20-100 range and which emit betas of energies up to ~8 MeV. These include all fission products and the actinides. The beta-energy spectra calculated by the present method are also of the same quality.

I. INTRODUCTION

The approximation discussed in this report was developed during studies of fission-product decay properties but its usefulness is not limited to fission products. Up until recently, because of lack of sufficient data, many fission-product decay-heat calculations were made by assigning some fixed fraction (~1/3) of the β^- endpoint energy as average β^- energy. With the increase in accuracy of recent decay-heat experiments, the degree in sophistication of theoretical calculations required to compare with experiments has made such ballpark estimates inadequate. Also, the continuous development and improvement of evaluated fission-product (ENDF/B-IV) data[1] has helped increase the quality of average decay-energy calculations.

One of the simplest, yet fairly accurate approximations used for computing average beta energies in fission-product calculations is given in Ref. (2). It was used in 1969 for data incorporated in the C INDER code[2,3] and subsequently for most ENDF/B-IV data.[1,4] Presented here is a superior method not only for use with fission products but also for actinides where the older method partially breaks down. The present method is quite accurate for the entire atomic number range from $\lesssim 20$ to $\gtrsim 100$ and for maximum beta energies up to ~ 8 MeV agreeing with "exact" calculations to within 1%. Also the spectra calculated by this method are very good. In the course of developing the present method, we have calculated beta spectra and beta average energies for all ENDF/B-IV fission-product nuclides using the exact equations discussed below.

II. THEORETICAL BACKGROUND

Beta-decay theory has been in continuous development since Fermi derived the first satisfactory quantum mechanical theory.[5] Therefore, a truly exact theory does not exist and what is referred to as exact in this report is the result of detailed quantum mechanical solutions to Dirac's equation for an electron in an electrostatic field. The simple expression derived below is an approximation to the best available theoretical results that can be used as a standard today and which are henceforth referred to as "exact."

The probability of beta emission with total relativistic energy, in electron rest energy units, between W and W + dW is[6,7]

$$N(Z,W) = C|M|^2 F(Z,W) K(W)\, W\sqrt{W^2-1}\,(W_0-W)^2, \quad (1)$$

where

$F(Z,W)$ = electron density ratio
C = a constant
$|M|^2$ = the square modulus of the transition matrix element
W_0 = maximum value of W
$K(W)$ = shape factor .

For allowed transitions, K(W) is equal to unity. For first, second, and third forbidden unique transitions, respectively, the shape factor is given by the following expressions[8]

$$K_1(W) = (W^2-1) + (W_0-W)^2 \quad , \quad (2)$$

$$K_2(W) = (W^2-1)^2 + (W_0-W)^4 + \frac{10}{3}(W^2-1)(W_0-W)^2 , \quad (3)$$

$$K_3(W) = (W^2-1)^3 + (W_0-W)^6 + 7\,(W^2-1)(W_0-W)^2\left[(W^2-1) + (W_0-W)^2\right] . \quad (4)$$

The electron density ratio, also referred to as the Fermi function F(Z,W), can be obtained by solving the non-relativistic (Schrödinger) or the relativistic (Dirac) equation for an electron in the electrostatic field of the nucleus. The Schrödinger equation for an electron in the point-charge field of a nucleus Ze^2/r yields the non-relativistic expression:

$$F_N(Z,W) = \frac{2\pi y}{1-e^{-2\pi y}} \quad , \quad (5)$$

where

$$y = \alpha Z \frac{W}{\sqrt{W^2-1}} \quad , \quad (6)$$

$$\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137} = \text{fine structure constant} \quad . \quad (7)$$

The corresponding relativistic expression for F(Z,W) is:

$$F_R(Z,W) = \frac{4(1+\frac{s}{2})}{|\Gamma(3+2s)|^2}\left(\frac{2R}{\lambda_c}\right)^{2s} e^{\pi y}\,(W^2-1)^s \times |\Gamma(1+s+iy)|^2 \quad , \quad (8)$$

$$s = \sqrt{1-(\alpha Z)^2} - 1 \quad , \quad (9)$$

$$\lambda_c = \frac{\hbar}{m_e c} \simeq 386 \times 10^{-13}\text{cm} \quad , \quad (10)$$

= the rationalized Compton wavelength of an electron.

A good expression for R, the nuclear radius, in terms of the mass number was given by Elton[9] as

$$R = (1.123A^{1/3} - 0.941A^{-1/3}) \times 10^{-13}\text{cm} \quad . \quad (11)$$

The spectrum-averaged beta kinetic energy is given by

$$\langle W-1\rangle = \frac{\int_1^{W_0} WN(W)dW}{\int_1^{W_0} N(W)dW} - 1 \quad . \quad (12)$$

F(Z,W), as given by Eq. (8), makes Eq. (12) integrable only by numerical techniques. For the sake of simplicity and expediency, it is desirable to have an analytically integrable form of F(Z,W) which produces the probability of the W distribution and average values of (W-1) in close agreement with those obtainable from Eq. (8). Such a form, mentioned above, has been used earlier in the CINDER code with success for average energies and also for ENDF/B-IV data.[1,4] In essence, this approximation uses a simplified non-relativistic expression for F(Z,W):

$$F(Z,W) \simeq 2\pi y \quad . \quad (13)$$

When Eqs. (13), (1), and (12) are combined, one obtains,[2] after some algebra:

$$\langle W-1\rangle = \frac{W_0^6 - 2W_0^5 + 5W_0^2 - 6W_0 + 2}{2(W_0^5 - 10W_0^2 + 15W_0 - 6)} \quad , \qquad (14)$$

or in terms of kinetic energy in units of electron rest energy,[1] X:

$$\langle X\rangle = X_0 \frac{X_0^2 + 4X_0 + 5}{2(X_0^2 + 5X_0 + 10)} \quad , \qquad (15)$$

where X_0 is the maximum value of X. We have found Eqs. (14) and (15) to be accurate for fission products to within 3% of the exact value but the average energies depart considerably from the exact values in the actinide region. Also, the spectra calculated with this method are not satisfactory.

Other approximations have also been investigated, like, for example, the Bethe-Bacher[10] approximation to $|\Gamma(1 + s + iy)|^2$:

$$|\Gamma(1 + s + iy)|^2 \simeq |\Gamma(1 + iy)|^2 \, (y^2 + \tfrac{1}{4})^s = \frac{\pi y}{\sinh \pi y} \left[(\alpha^2 Z^2 + \tfrac{1}{4}) W^2 - \tfrac{1}{4} \right]^s \quad . \qquad (16)$$

This expression avoids complex gamma function calculations but still requires numerical integration of Eq. (12).

Another approximate expression of F(Z,W) which avoids complex gamma function calculations but still requires numerical integration of Eq. (12) was given by Hall:[11]

$$F(Z,W) = \frac{4(1 + \frac{s}{2})}{|\Gamma(3 + 2s)|^2} \left(\frac{2R}{\lambda_c}\right)^{2s} \times \left[(s + 1)^2 + y^2\right]^{s + \frac{1}{2}} \times \exp \left\{ 2\phi y - 2(s + 1) + \frac{s + 1}{6\left[(s + 1)^2 + y^2\right]} \right\}, \qquad (17)$$

where

$$\phi = \arctan \left(\frac{s + 1}{y} \right) . \qquad (18)$$

III. A NEW APPROXIMATE METHOD

After numerous attempts with the various available approximations, we have found that we can approximate the following product of mathmatically complicated functions by a second order polynomial, i.e.,

$$F(Z,W) \sqrt{W^2 - 1} \left(\frac{2R}{\lambda_c}\right)^{-2s} \simeq A_0(Z) + A_1(Z)W + A_2(Z)W^2 \quad . \qquad (19)$$

In terms of this approximation, Eq. (1) becomes

$$N(W) = C|M|^2 \left(\frac{2R}{\lambda_c}\right)^{2s} n(W) \quad , \qquad (20)$$

where

$$n(W) = (A_0 + A_1 W + A_2 W^2) \; W \; (W_0 - W)^2 K(W) . \qquad (21)$$

We can now write the average kinetic beta energy as

$$\langle W - 1\rangle = \frac{\int_1^{W_0} W n(W) dW}{\int_1^{W_0} n(W) dW} - 1 \quad , \qquad (22)$$

which can be analytically integrated to give for an allowed transition,

$$\langle W-1\rangle = \frac{\sum_{i=0}^{7} B_i W_0^i}{\sum_{i=0}^{6} C_i W_0^i} \quad , \qquad (23)$$

where

$$B_0 = \frac{A_0}{20} + \frac{A_1}{30} + \frac{A_2}{42} \quad , \qquad (24)$$

$$B_1 = -\left(\frac{A_0}{6} + \frac{A_1}{10} + \frac{A_2}{15}\right) \quad , \qquad (25)$$

$$B_2 = \frac{A_0}{6} + \frac{A_1}{12} + \frac{A_2}{20} \quad , \tag{26}$$

$$B_3 = 0 \quad , \tag{27}$$

$$B_4 = -\frac{A_0}{12} \quad , \tag{28}$$

$$B_5 = \frac{A_0 - A_1}{30} \quad , \tag{29}$$

$$B_6 = \frac{A_1 - A_2}{60} \quad , \tag{30}$$

$$B_7 = \frac{A_2}{105} \quad , \tag{31}$$

$$C_0 = -\left(\frac{A_0}{4} + \frac{A_1}{5} + \frac{A_2}{6}\right) \quad , \tag{32}$$

$$C_1 = \frac{2A_0}{3} + \frac{A_1}{2} + \frac{2A_2}{5} \quad , \tag{33}$$

$$C_2 = -\left(\frac{A_0}{2} + \frac{A_1}{3} + \frac{A_2}{4}\right) \quad , \tag{34}$$

$$C_3 = 0 \quad , \tag{35}$$

$$C_4 = \frac{A_0}{12} \quad , \tag{36}$$

$$C_5 = \frac{A_1}{30} \quad , \tag{37}$$

$$C_6 = \frac{A_2}{60} \quad . \tag{38}$$

When $A_0 = A_2 = 0$ and $A_1 = 1$, Eq. (23) reduces, as expected, to Eq. (14).

Alternately, Eq. (23) can be expressed in terms of kinetic energy in $m_e c^2$ units:

$$\langle X \rangle = X_0 \frac{\sum_{i=0}^{3} Q_i X_0^{\,i}}{\sum_{i=0}^{3} P_i X_0^{\,i}} \quad , \tag{39}$$

where

$$Q_0 = \frac{A_0 + A_1 + A_2}{12} \quad , \tag{40}$$

$$Q_1 = \frac{A_0}{30} + \frac{A_1}{15} + \frac{A_2}{10} \quad , \tag{41}$$

$$Q_2 = \frac{A_1}{60} + \frac{A_2}{20} \quad , \tag{42}$$

$$Q_3 = \frac{A_2}{105} \quad , \tag{43}$$

$$P_0 = \frac{A_0 + A_1 + A_2}{3} \quad , \tag{44}$$

$$P_1 = \frac{A_0}{12} + \frac{A_1}{6} + \frac{A_2}{4} \quad , \tag{45}$$

$$P_2 = \frac{A_1}{30} + \frac{A_2}{10} \quad , \tag{46}$$

$$P_3 = \frac{A_2}{60} \quad . \tag{47}$$

Further simplification in terms of the number of coefficients required results when (X) is written in the following form

$$\langle X \rangle = X_0 \frac{5a_0 + 4a_1 X_0 + a_2 X_0^{\,2} + 8a_3 X_0^{\,3}}{2\left(10a_0 + 5a_1 X_0 + a_2 X_0^{\,2} + 7a_3 X_0^{\,3}\right)} \quad , \tag{48}$$

where

$$a_0 = A_0 + A_1 + A_2 \quad , \tag{49}$$

$$a_1 = \frac{A_0}{2} + A_1 + \frac{3A_2}{2} \quad , \tag{50}$$

$$a_2 = A_1 + 3A_2 \quad , \tag{51}$$

$$a_3 = \frac{A_2}{12} \quad . \tag{52}$$

Actually, a_3 is a redundant coefficient because

$$a_3 = \frac{a_0 + a_2 - 2a_1}{12} \quad . \tag{53}$$

Eq. (48) can be easily calculated even on a pocket calculator provided the a_i coefficients or the A_i coefficients are known as a function of the atomic number Z.

The coefficients A_0, A_1, and A_2 were least-squares fitted as a function of Z and the results are shown in Table I. Coefficients a_0, a_1, a_2, and a_3 can, of course, be calculated from A_0, A_1, and A_2 using Eqs. (49) to (52). The results are given, for convenience, in Table II.

TABLE I
"A" COEFFICIENTS Vs. Z

A0	A1	A2	Z
-.2156E+01	.2367E+01	-.1363E+00	1
-.2108E+01	.2366E+01	-.1351E+00	2
-.1983E+01	.2184E+01	-.1105E+00	3
-.1859E+01	.2183E+01	-.1086E+00	4
-.1765E+01	.2127E+01	-.9955E-01	5
-.1676E+01	.2074E+01	-.9074E-01	6
-.1600E+01	.2037E+01	-.8383E-01	7
-.1557E+01	.2037E+01	-.8133E-01	8
-.1475E+01	.1995E+01	-.7365E-01	9
-.1432E+01	.1995E+01	-.7114E-01	10
-.1366E+01	.1971E+01	-.6552E-01	11
-.1322E+01	.1972E+01	-.6297E-01	12
-.1277E+01	.1972E+01	-.6033E-01	13
-.1232E+01	.1973E+01	-.5763E-01	14
-.1186E+01	.1974E+01	-.5490E-01	15
-.1140E+01	.1975E+01	-.5216E-01	16
-.1093E+01	.1977E+01	-.4943E-01	17
-.1045E+01	.1979E+01	-.4676E-01	18
-.9977E+00	.1983E+01	-.4410E-01	19
-.9592E+00	.1996E+01	-.4269E-01	20
-.9106E+00	.2000E+01	-.4013E-01	21
-.8753E+00	.2020E+01	-.3934E-01	22
-.8373E+00	.2038E+01	-.3832E-01	23
-.7971E+00	.2054E+01	-.3714E-01	24
-.7549E+00	.2070E+01	-.3583E-01	25
-.7150E+00	.2089E+01	-.3492E-01	26
-.6747E+00	.2110E+01	-.3411E-01	27
-.6343E+00	.2131E+01	-.3340E-01	28
-.5935E+00	.2154E+01	-.3279E-01	29
-.5525E+00	.2178E+01	-.3231E-01	30
-.5109E+00	.2202E+01	-.3193E-01	31
-.4679E+00	.2227E+01	-.3153E-01	32
-.4254E+00	.2254E+01	-.3138E-01	33
-.3823E+00	.2282E+01	-.3132E-01	34
-.3384E+00	.2311E+01	-.3137E-01	35
-.2938E+00	.2340E+01	-.3152E-01	36
-.2483E+00	.2371E+01	-.3176E-01	37
-.2019E+00	.2402E+01	-.3211E-01	38
-.1546E+00	.2435E+01	-.3256E-01	39
-.1061E+00	.2468E+01	-.3311E-01	40
-.5657E-01	.2501E+01	-.3376E-01	41
-.5782E-02	.2536E+01	-.3451E-01	42
.4630E-01	.2571E+01	-.3536E-01	43
.9978E-01	.2607E+01	-.3631E-01	44
.1547E+00	.2644E+01	-.3735E-01	45
.2112E+00	.2681E+01	-.3850E-01	46
.2694E+00	.2719E+01	-.3974E-01	47
.3293E+00	.2758E+01	-.4108E-01	48
.3910E+00	.2797E+01	-.4252E-01	49
.4547E+00	.2836E+01	-.4405E-01	50
.5205E+00	.2876E+01	-.4567E-01	51
.5883E+00	.2916E+01	-.4739E-01	52
.6585E+00	.2957E+01	-.4920E-01	53
.7311E+00	.2997E+01	-.5110E-01	54
.8058E+00	.3038E+01	-.5307E-01	55
.8835E+00	.3079E+01	-.5515E-01	56
.9641E+00	.3121E+01	-.5731E-01	57
.1047E+01	.3162E+01	-.5956E-01	58
.1134E+01	.3203E+01	-.6188E-01	59
.1224E+01	.3244E+01	-.6428E-01	60
.1317E+01	.3285E+01	-.6676E-01	61
.1413E+01	.3326E+01	-.6930E-01	62
.1513E+01	.3366E+01	-.7191E-01	63
.1617E+01	.3405E+01	-.7459E-01	64
.1725E+01	.3445E+01	-.7732E-01	65
.1837E+01	.3483E+01	-.8010E-01	66
.1953E+01	.3521E+01	-.8294E-01	67
.2074E+01	.3558E+01	-.8581E-01	68
.2200E+01	.3593E+01	-.8872E-01	69
.2330E+01	.3628E+01	-.9166E-01	70
.2466E+01	.3661E+01	-.9459E-01	71
.2607E+01	.3693E+01	-.9758E-01	72
.2753E+01	.3724E+01	-.1006E+00	73
.2906E+01	.3753E+01	-.1036E+00	74
.3064E+01	.3780E+01	-.1065E+00	75
.3228E+01	.3805E+01	-.1095E+00	76
.3399E+01	.3828E+01	-.1124E+00	77
.3576E+01	.3848E+01	-.1153E+00	78
.3761E+01	.3866E+01	-.1181E+00	79
.3953E+01	.3881E+01	-.1209E+00	80
.4152E+01	.3893E+01	-.1236E+00	81
.4359E+01	.3903E+01	-.1261E+00	82
.4573E+01	.3909E+01	-.1286E+00	83
.4796E+01	.3911E+01	-.1310E+00	84
.5035E+01	.3903E+01	-.1325E+00	85
.5268E+01	.3904E+01	-.1353E+00	86
.5519E+01	.3893E+01	-.1370E+00	87
.5776E+01	.3879E+01	-.1388E+00	88
.6041E+01	.3864E+01	-.1407E+00	89
.6321E+01	.3839E+01	-.1419E+00	90
.6609E+01	.3810E+01	-.1430E+00	91
.6907E+01	.3776E+01	-.1438E+00	92
.7217E+01	.3735E+01	-.1443E+00	93
.7536E+01	.3689E+01	-.1445E+00	94
.7867E+01	.3636E+01	-.1444E+00	95
.8209E+01	.3577E+01	-.1440E+00	96
.8563E+01	.3511E+01	-.1432E+00	97
.8928E+01	.3438E+01	-.1420E+00	98
.9304E+01	.3358E+01	-.1405E+00	99
.9694E+01	.3270E+01	-.1385E+00	100
.1009E+02	.3174E+01	-.1360E+00	101

TABLE II

"a" COEFFICIENTS Vs. Z

a_0	a_1	a_2	a_3	Z
7.4700E-02	1.0846E+00	1.9581E+00	-9.7357E-03	1
1.2290E-01	1.1095E+00	1.9607E+00	-9.6500E-03	2
1.7050E-01	1.0667E+00	1.8525E+00	-7.8929E-03	3
2.1540E-01	1.0906E+00	1.8572E+00	-7.7571E-03	4
2.6245E-01	1.0952E+00	1.8283E+00	-7.1107E-03	5
3.0726E-01	1.0999E+00	1.8018E+00	-6.4814E-03	6
3.5317E-01	1.1113E+00	1.7855E+00	-5.9879E-03	7
3.9867E-01	1.1365E+00	1.7930E+00	-5.8093E-03	8
4.4635E-01	1.1470E+00	1.7740E+00	-5.2607E-03	9
4.9186E-01	1.1723E+00	1.7816E+00	-5.0814E-03	10
5.3948E-01	1.1897E+00	1.7744E+00	-4.6800E-03	11
5.8703E-01	1.2165E+00	1.7831E+00	-4.4979E-03	12
6.3467E-01	1.2430E+00	1.7910E+00	-4.3093E-03	13
6.8337E-01	1.2706E+00	1.8001E+00	-4.1164E-03	14
7.3310E-01	1.2986E+00	1.8093E+00	-3.9214E-03	15
7.8284E-01	1.3268E+00	1.8185E+00	-3.7257E-03	16
8.3457E-01	1.3564E+00	1.8287E+00	-3.5397E-03	17
8.8724E-01	1.3864E+00	1.8387E+00	-3.3400E-03	18
9.4120E-01	1.4180E+00	1.8507E+00	-3.1500E-03	19
9.9411E-01	1.4524E+00	1.8679E+00	-3.0493E-03	20
1.0493E+00	1.4845E+00	1.8796E+00	-2.8664E-03	21
1.1054E+00	1.5233E+00	1.9020E+00	-2.8100E-03	22
1.1624E+00	1.5619E+00	1.9230E+00	-2.7371E-03	23
1.2198E+00	1.5997E+00	1.9426E+00	-2.6529E-03	24
1.2793E+00	1.6388E+00	1.9625E+00	-2.5593E-03	25
1.3391E+00	1.6791E+00	1.9842E+00	-2.4943E-03	26
1.4012E+00	1.7215E+00	2.0077E+00	-2.4364E-03	27
1.4633E+00	1.7637E+00	2.0308E+00	-2.3857E-03	28
1.5277E+00	1.8081E+00	2.0556E+00	-2.3421E-03	29
1.5932E+00	1.8533E+00	2.0811E+00	-2.3079E-03	30
1.6592E+00	1.8987E+00	2.1062E+00	-2.2807E-03	31
1.7276E+00	1.9458E+00	2.1324E+00	-2.2521E-03	32
1.7972E+00	1.9942E+00	2.1599E+00	-2.2414E-03	33
1.8684E+00	2.0439E+00	2.1880E+00	-2.2371E-03	34
1.9412E+00	2.0947E+00	2.2169E+00	-2.2407E-03	35
2.0147E+00	2.1458E+00	2.2454E+00	-2.2514E-03	36
2.0909E+00	2.1992E+00	2.2757E+00	-2.2686E-03	37
2.1680E+00	2.2529E+00	2.3057E+00	-2.2936E-03	38
2.2478E+00	2.3089E+00	2.3373E+00	-2.3257E-03	39
2.3288E+00	2.3653E+00	2.3687E+00	-2.3650E-03	40
2.4107E+00	2.4221E+00	2.3997E+00	-2.4114E-03	41
2.4957E+00	2.4813E+00	2.4325E+00	-2.4650E-03	42
2.5819E+00	2.5411E+00	2.4649E+00	-2.5257E-03	43
2.6705E+00	2.6024E+00	2.4981E+00	-2.5936E-03	44
2.7613E+00	2.6653E+00	2.5319E+00	-2.6679E-03	45
2.8537E+00	2.7288E+00	2.5655E+00	-2.7500E-03	46
2.9487E+00	2.7941E+00	2.5998E+00	-2.8386E-03	47
3.0462E+00	2.8610E+00	2.6348E+00	-2.9343E-03	48
3.1455E+00	2.9287E+00	2.6694E+00	-3.0371E-03	49
3.2466E+00	2.9973E+00	2.7038E+00	-3.1464E-03	50
3.3508E+00	3.0677E+00	2.7390E+00	-3.2621E-03	51
3.4569E+00	3.1391E+00	2.7738E+00	-3.3850E-03	52
3.5663E+00	3.2124E+00	2.8094E+00	-3.5143E-03	53
3.6770E+00	3.2859E+00	2.8437E+00	-3.6500E-03	54
3.7907E+00	3.3613E+00	2.8788E+00	-3.7907E-03	55
3.9073E+00	3.4380E+00	2.9135E+00	-3.9393E-03	56
4.0276E+00	3.5171E+00	2.9491E+00	-4.0936E-03	57
4.1494E+00	3.5962E+00	2.9833E+00	-4.2543E-03	58
4.2751E+00	3.6772E+00	3.0174E+00	-4.4200E-03	59
4.4037E+00	3.7596E+00	3.0512E+00	-4.5914E-03	60
4.5352E+00	3.8434E+00	3.0847E+00	-4.7686E-03	61
4.6697E+00	3.9285E+00	3.1181E+00	-4.9500E-03	62
4.8071E+00	4.0146E+00	3.1503E+00	-5.1364E-03	63
4.9474E+00	4.1016E+00	3.1812E+00	-5.3279E-03	64
5.0927E+00	4.1915E+00	3.2130E+00	-5.5229E-03	65
5.2399E+00	4.2813E+00	3.2427E+00	-5.7214E-03	66

TABLE II
(CONTINUATION)

a_0	a_1	a_2	a_3	Z
5.3911E+00	4.3731E+00	3.2722E+00	-5.9243E-03	67
5.5462E+00	4.4663E+00	3.3006E+00	-6.1293E-03	68
5.7043E+00	4.5599E+00	3.3268E+00	-6.3371E-03	69
5.8663E+00	4.6555E+00	3.3530E+00	-6.5471E-03	70
6.0324E+00	4.7521E+00	3.3772E+00	-6.7564E-03	71
6.2024E+00	4.8501E+00	3.4003E+00	-6.9700E-03	72
6.3764E+00	4.9496E+00	3.4222E+00	-7.1857E-03	73
6.5554E+00	5.0506E+00	3.4422E+00	-7.4000E-03	74
6.7375E+00	5.1522E+00	3.4605E+00	-7.6071E-03	75
6.9235E+00	5.2547E+00	3.4765E+00	-7.8214E-03	76
7.1146E+00	5.3589E+00	3.4908E+00	-8.0286E-03	77
7.3087E+00	5.4630E+00	3.5021E+00	-8.2357E-03	78
7.5089E+00	5.5693E+00	3.5117E+00	-8.4357E-03	79
7.7131E+00	5.6761E+00	3.5183E+00	-8.6357E-03	80
7.9214E+00	5.7836E+00	3.5222E+00	-8.8286E-03	81
8.1359E+00	5.8933E+00	3.5247E+00	-9.0071E-03	82
8.3534E+00	6.0026E+00	3.5232E+00	-9.1857E-03	83
8.5760E+00	6.1125E+00	3.5180E+00	-9.3571E-03	84
8.8055E+00	6.2217E+00	3.5055E+00	-9.4643E-03	85
9.0367E+00	6.3350E+00	3.4981E+00	-9.6643E-03	86
9.2750E+00	6.4470E+00	3.4820E+00	-9.7857E-03	87
9.5162E+00	6.5588E+00	3.4626E+00	-9.9143E-03	88
9.7643E+00	6.6735E+00	3.4419E+00	-1.0050E-02	89
1.0018E+01	6.7866E+00	3.4133E+00	-1.0136E-02	90
1.0276E+01	6.9000E+00	3.3810E+00	-1.0214E-02	91
1.0539E+01	7.0138E+00	3.3446E+00	-1.0271E-02	92
1.0808E+01	7.1270E+00	3.3021E+00	-1.0307E-02	93
1.1080E+01	7.2402E+00	3.2555E+00	-1.0321E-02	94
1.1359E+01	7.3529E+00	3.2028E+00	-1.0314E-02	95
1.1642E+01	7.4655E+00	3.1450E+00	-1.0286E-02	96
1.1931E+01	7.5777E+00	3.0814E+00	-1.0229E-02	97
1.2224E+01	7.6890E+00	3.0120E+00	-1.0143E-02	98
1.2521E+01	7.7992E+00	2.9365E+00	-1.0036E-02	99
1.2825E+01	7.9092E+00	2.8545E+00	-9.8929E-03	100
1.3128E+01	8.0150E+00	2.7660E+00	-9.7143E-03	101

Finally, coefficients a_0, a_1, and a_2 were approximated by least-squares fitting to explicit functions of Z:

$$a_0 = \sum_{i=0}^{4} a_{0i} Z^i \quad , \tag{54}$$

$$a_1 = \sum_{i=0}^{5} a_{1i} Z^i \quad , \tag{55}$$

$$a_2 = a_{20} + a_{21} Z + a_{22} Z^2 + a_{23} \exp\left[-a_{24}(Z - a_{25})^2\right] , \tag{56}$$

where coefficients a_{00} through a_{25} are given in Table III. Coefficient a_3 can be calculated from Eq. (53) in terms of a_0, a_1, and a_2.

For forbidden unique transitions, the average kinetic beta energy can again be calculated from equations like (23) or (39) in terms of coefficients A_0, A_1, and A_2 or a_0, a_1, and a_2. For example, for first forbidden unique transitions, the average kinetic beta energy can be calculated from Eqs. (1), (2), and (22) and the result, after a good deal of algebra, is

$$\langle x \rangle = X_0 \frac{\sum_{i=0}^{4} p_i X_0^i}{\sum_{i=0}^{4} q_i X_0^i} \quad , \tag{57}$$

TABLE III

COEFFICIENTS FOR EQUATIONS 54 - 56

i	a_{0i}	a_{1i}	a_{2i}
0	3.3321E-02	9.6243E-01	1.3837E+00
1	4.4126E-02	1.5766E-02	-6.9268E-03
2	9.3870E-05	4.3264E-04	-3.8474E-04
3	4.8527E-06	1.1359E-07	6.0824E+00
4	2.6034E-08	3.8335E-08	2.8856E-04
5	—	-2.8872E-10	1.0401E+02

where

$$p_0 = \frac{A_0 + A_1 + A_2}{15} \quad , \tag{58}$$

$$p_1 = \frac{1}{60}(5A_0 + 7A_1 + 9A_2) \quad , \tag{59}$$

$$p_2 = \frac{2}{105}(A_0 + 3A_1 + 6A_2) \quad , \tag{60}$$

$$p_3 = \frac{1}{105}(A_1 + \frac{17}{4}A_2) \quad , \tag{61}$$

$$p_4 = \frac{A_2}{180} \quad , \tag{62}$$

$$q_0 = \frac{A_0 + A_1 + A_2}{6} \quad , \tag{63}$$

$$q_1 = \frac{1}{30}(9A_0 + 11A_1 + 13A_2) \quad , \tag{64}$$

$$q_2 = \frac{1}{60}(3A_0 + 8A_1 + 15A_2) \quad , \tag{65}$$

$$q_3 = \frac{2}{105}(A_1 + 4A_2) \quad , \tag{66}$$

$$q_4 = \frac{A_2}{105} \quad . \tag{67}$$

IV. BETA SPECTRA

The beta spectrum in a multigroup form for a specific nuclide is given by:

$$N_i = E_0 \int_{X_i}^{X_{i+1}} N(X)dX,$$

$$i = 0, M \quad , \tag{68}$$

where X_i is the i-th group boundary kinetic energy in electron rest energy units E_0, and M is the number of groups. The spectrum can be constructed either by numerically integrating Eq. (68) M times with the "exact" value of N(X) or by analytically integrating Eq. (68) with the approximate value of N(X) as given by Eqs. (20) and (21). The cumulative group spectrum of nuclide mixtures is given by:

$$S_i = E_0 \sum_{j=1}^{J} C_j \int_{X_i}^{X_{i+1}} N_j(X)dX,$$

$$i = 0, M \quad , \tag{69}$$

where subscript j is the index of a specific nuclide having atomic concentration C_j and J is the total number of nuclides involved.

V. COMPARISONS AND DISCUSSION

The results obtained by the new method, Eq. (48), were compared with results obtained by numerically integrating Eq. (12) with F(Z,W) given by Eq. (8). The latter "exact" method has required complex gamma function calculating routines and accurate numerical integration techniques. The numerical integrations were performed with a variable-grid Simpson routine permitting any

TABLE IV

AVERAGE BETA ENERGY COMPARISONS OF THE EXACT AND APPROXIMATE METHOD

Z \ Max. β^- Energy	10^4 eV			10^5 eV			10^6 eV		
	Exact	Eq. (48)	% Diff.	Exact	Eq. (48)	% Diff.	Exact	Eq. (48)	% Diff.
21	2.514+03	2.521+03	0.251	2.720+04	2.697+04	0.851	3.703+05	3.731+05	0.754
26	2.513+03	2.518+03	0.196	2.681+04	2.676+04	0.202	3.653+05	3.661+05	0.219
31	2.513+03	2.517+03	0.143	2.658+04	2.661+04	0.130	3.609+05	3.605+05	0.115
36	2.513+03	2.516+03	0.103	2.643+04	2.651+04	0.271	3.569+05	3.558+05	0.299
41	2.513+03	2.515+03	0.073	2.634+04	2.642+04	0.304	3.533+05	3.519+05	0.372
46	2.513+03	2.514+03	0.051	2.628+04	2.636+04	0.278	3.499+05	3.486+05	0.371
51	2.513+03	2.513+03	0.034	2.624+04	2.630+04	0.224	3.468+05	3.456+05	0.327
56	2.512+03	2.513+03	0.020	2.621+04	2.625+04	0.157	3.438+05	3.429+05	0.270
61	2.512+03	2.512+03	0.009	2.618+04	2.621+04	0.089	3.409+05	3.402+05	0.219
66	2.512+03	2.512+03	0.000	2.616+04	2.616+04	0.023	3.381+05	3.375+05	0.191
71	2.512+03	2.512+03	0.007	2.613+04	2.612+04	0.038	3.353+05	3.347+05	0.193
76	2.511+03	2.511+03	0.013	2.610+04	2.610+04	0.092	3.325+05	3.318+05	0.227
81	2.511+03	2.511+03	0.018	2.607+04	2.604+04	0.137	3.296+05	3.287+05	0.284
86	2.511+03	2.510+03	0.021	2.604+04	2.600+04	0.173	3.266+05	3.254+05	0.350
91	2.510+03	2.510+03	0.024	2.600+04	2.595+04	0.197	3.233+05	3.221+05	0.398
96	2.510+03	2.510+03	0.025	2.596+04	2.591+04	0.208	3.200+05	3.186+05	0.397

arbitrary user-required accuracy. This method has proved quite tedious and very time-consuming by comparison with Eq. (48) which can be easily implemented with a pocket calculator. When Eq. (48) was used with the coefficients given either by Tables I or II, or by Eqs. (54)-(56) and Table III, the calculated average kinetic beta energies were found to be well within 1% of the exact values for nuclides with atomic numbers in the range $\lesssim$20 to $\gtrsim$100 and for maximum kinetic beta energies of up to ~8 MeV (see Table IV). This range includes all fission-product nuclides and the actinides. The spectra calculated by this method are also very good.

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